

SHARP LOWER BOUND ESTIMATES FOR VECTOR-VALUED AND MATRIX-VALUED MULTIPLIERS IN L^p

NICHOLAS BOROS AND ALEXANDER VOLBERG

ABSTRACT. We generalize the idea of a multiplier in two different ways. First of all, we consider multipliers in the form of a vector acting on a scalar function. Using this technique we are able to show that for $\tau^2 \leq p^* - 1$ and $1 < p < 2$ or $\tau \in \mathbb{R}$ and $2 \leq p < \infty$, $\left\| \begin{pmatrix} Re B \\ \tau I \end{pmatrix} \right\|_{L^p(\mathbb{C}, \mathbb{C}) \rightarrow L^p(\mathbb{C}, \mathbb{C}^2)} \geq ((p^* - 1)^2 + \tau^2)^{\frac{1}{2}}$, where B is the

Ahlfors–Beurling operator, I is the identity operator and $p^* - 1 = \max\left\{p - 1, \frac{1}{p-1}\right\}$. Secondly, we consider matrix-valued multipliers to obtain a new proof showing that $\|B\|_{L^p(\mathbb{C}) \rightarrow L^p(\mathbb{C})} \geq p^* - 1$.

1. Introduction

The results of Geiss, Montgomery-Smith, Saksman [7] generalized from an idea of Bourgain [4] seems to be quite robust. Bourgain originally used this technique for the Hilbert transform, to get an L^p lower bound in terms of UMD constant. Geiss, Montgomery-Smith, Saksman were able to generalize this quite a bit further to the class of real-valued, even multipliers that are continuous and homogenous of order zero. The argument, with a little bit of effort will generalize even further. We are able to get a lower bound of an operator corresponding to either vector-valued multipliers or matrix-valued multipliers also in terms of the natural generalization of UMD . With matrix valued multipliers, one can get the lower bound of an L^p operator corresponding to a complex multiplier in terms of UMD . We present the Ahlfors–Beurling operator as such an example.

2. Vector-valued Multipliers

2.1. Definitions and Notation. Let $1 \leq p, p_0 \leq \infty$ and $m : \mathbb{R}^n \rightarrow \mathbb{C}$ be bounded function. We say that m is the multiplier corresponding with the operator T_m , if $T_m f := (m\widehat{f})^\vee$ is a bounded operator from $L^p(\mathbb{R}^n) \rightarrow L^{p_0}(\mathbb{R}^n)$, considering functions $f \in \mathcal{S}$. Where $\widehat{}$ and $^\vee$ are the Fourier transform and its inverse and \mathcal{S}

is the Schwartz class. The space of all such m is denoted $\mathcal{M}_{p,p_0}(\mathbb{R}^n)$ with norm $\|m\|_{\mathcal{M}_{p,p_0}(\mathbb{R}^n)} := \|T_m\|_{L^p \rightarrow L^{p_0}}$.

Similarly, for $1 \leq p, p_0 \leq \infty$ we can consider a bounded sequence $a = \{a_k\}_{k \in \mathbb{Z}^n} \subset \mathbb{C}$. We say that a is the discrete multiplier corresponding the operator T_a , if

$$(T_a f)(\theta) := (a\widehat{f})^\vee(\theta) = \sum_{k \in \mathbb{Z}^n} a_k \widehat{f}(k) e^{i(k, \theta)}$$

is a bounded operator from $L^p(\mathbb{T}^n) \rightarrow L^{p_0}(\mathbb{T}^n)$, considering functions f being trigonometric polynomials on $\mathbb{T}^n := [-\pi, \pi]^n$. The space of all such a is denoted $\mathcal{M}_{p,p_0}(\mathbb{Z}^n)$ with norm $\|a\|_{\mathcal{M}_{p,p_0}(\mathbb{Z}^n)} := \|T_m\|_{L^p \rightarrow L^{p_0}}$. In this paper, we will just take $a_k = m(k)$ and call \widetilde{m} the discrete multiplier under this restriction so that the operator is defined as $(T_{\widetilde{m}} f)(\theta) = \sum_{k \in \mathbb{Z}^n} m(k) \widehat{f}(k) e^{i(k, \theta)}$.

Generalizing the idea of a multiplier, let us consider $M = \{m_i\}_{1 \leq i \leq m}$ as a vector-valued function whose entries are bounded functions from \mathbb{R}^d to \mathbb{C} and p, p_0 such that $1 \leq p, p_0 \leq \infty$. We say that M is the multiplier corresponding to the operator T_M , if $T_M f := (M\widehat{f})^\vee$ (the inverse Fourier transform defined component-wise) is a bounded operator from $L^p(\mathbb{R}^d, \mathbb{C})$ to $L^{p_0}(\mathbb{R}^d, \mathbb{C}^m)$, considering functions f from \mathbb{R}^d to \mathbb{C} in \mathcal{S} . Note that $\widehat{T_M f} = \widehat{f} M$ as an extension of the scalar case and $\|f\|_{L^{p_0}(\mathbb{R}^d, \mathbb{C}^m)}^{p_0} := \int_{\mathbb{R}^d} \|f(x)\|_{\mathbb{C}^m}^{p_0} dx$. We use the notation $\widetilde{M} = (\widetilde{m}_i)_{1 \leq i \leq m}$ to denote the discrete multiplier.

Remark 1. We will use the notation $\mathcal{V}_m^{p,p_0}(\mathbb{R}^d)$ to denote the class of all continuous (except possibly at zero) and homogeneous of order zero vector multipliers, where $\|M\|_{\mathcal{V}_m^{p,p_0}} := \|T_M\|_{L^p(\mathbb{R}^d, \mathbb{C}) \rightarrow L^{p_0}(\mathbb{R}^d, \mathbb{C}^m)}$ for any such $M \in \mathcal{V}_m^{p,p_0}(\mathbb{R}^d)$.

2.2. Operator Norm of a Quadratic Perturbation of Martingale Transform. Let $\{r_n\}_{n \geq 0}$ the Rademacher sequence. Let $\{F_N\}_{N \geq 0}$ and $\{G_N\}_{N \geq 0}$ be \mathbb{C} -valued martingale difference sequences of the form

$$F_N = \sum_{k=1}^N d_k(r_0, \dots, r_{k-1}) r_k, \quad G_N = \sum_{k=1}^N \beta_k d_k(r_0, \dots, r_{k-1}) r_k$$

where $d_k : \{\pm 1\}^k \rightarrow \mathbb{C}$ is integrable and $\vec{\beta}$ is a vector with entries $\beta_i \in \pm 1$. For any particular such $\vec{\beta}$, $MT_{\vec{\beta}}(F_N) := G_N$ is a martingale transform. Let $\tau \in \mathbb{R}$ and

$\begin{pmatrix} MT_{\vec{\beta}} \\ \tau I \end{pmatrix} : L^p(\mathbb{C}) \rightarrow L^{p_0}(\mathbb{C}^2)$ be defined as $F_N \mapsto \begin{pmatrix} G_N \\ \tau F_N \end{pmatrix}$. Then

$$\left\| \begin{pmatrix} MT_{\vec{\beta}} \\ \tau I \end{pmatrix} \right\|_{L^p(\mathbb{C}) \rightarrow L^{p_0}(\mathbb{C}^2)} = \sup_{\vec{\beta}} \frac{\left\| \sum_{k=1}^N \begin{pmatrix} \beta_k \\ \tau \end{pmatrix} d_k(r_0, \dots, r_{k-1}) r_k \right\|_{p_0}}{\left\| \sum_{k=1}^N d_k(r_0, \dots, r_{k-1}) r_k \right\|_p}. \quad (2.1)$$

Definition 2. $C_{p,p_0}^\tau := \left\| \begin{pmatrix} MT_{\vec{\beta}} \\ \tau I \end{pmatrix} \right\|_{L^p(\mathbb{C}) \rightarrow L^{p_0}(\mathbb{C}^2)}.$

Note that $C_{p,p}^\tau = ((p^* - 1)^2 + \tau^2)^{\frac{1}{2}}$ for $\tau^2 \leq p^* - 1$ and $1 < p < 2$ or $\tau \in \mathbb{R}$ and $2 \leq p < \infty$ and shown in [1], [2] and [3], where $p^* = \max\{p, \frac{p}{p-1}\}$.

2.3. Transference. The following results are called transference properties of multipliers. The main result that we will prove is a lower bound for $\|T_M\|_{L^p(\mathbb{R}) \rightarrow L^{p_0}(\mathbb{R}^d)}$. Transference allows us to reduce this to getting a lower bound for $\|T_M\|_{L^p(\mathbb{T}) \rightarrow L^{p_0}(\mathbb{T}^d)}$, which is easier to work with.

Lemma 3. Suppose $1 < p_0 \leq p < \infty$ and $M \in \mathcal{V}_m^{p,p_0}(\mathbb{R}^d)$, then $\|T_M\|_{L^p(\mathbb{R}^d, \mathbb{C}) \rightarrow L^{p_0}(\mathbb{R}^d, \mathbb{C}^m)} \geq \|T_{\tilde{M}}\|_{L^p(\mathbb{T}^d, \mathbb{C}) \rightarrow L^{p_0}(\mathbb{T}^d, \mathbb{C}^m)}.$

Proof. The proof that we present here is a generalization of the proof in [8], on pp. 221–223, for m being a scalar valued multiplier, so we will briefly describe the difference in the argument needed. We claim that for P and Q being trigonometric polynomials, $L_\varepsilon(x) = e^{-\pi\varepsilon|x|^2}$ and $\frac{1}{p_0} + \frac{1}{q_0} = 1$ we have

$$\int_{\mathbb{T}^d} ((T_{\tilde{M}}P)(x), Q(x)) dx = \lim_{\varepsilon \rightarrow 0} \varepsilon^{\frac{n}{2}} \int_{\mathbb{R}^n} (T_M(PL_{\frac{\varepsilon}{p_0}})(x), Q(x)L_{\frac{\varepsilon}{q_0}}(x)) dx.$$

Indeed, by linearity we can just consider $P(x) = e^{2\pi i(j,x)}$ and $Q(x) = e^{2\pi i(k,x)}b$, where $b \in \mathbb{C}^m$. By Parseval's identity we have that $\int_{\mathbb{T}^d} (T_{\tilde{M}}P(x), Q(x))_{\mathbb{C}^m} dx = (M(j), b)_{\mathbb{C}^m} \delta_{jk}$. On the other hand

$$\begin{aligned} & \varepsilon^{\frac{d}{2}} \int_{\mathbb{R}^d} (T_M(PL_{\frac{\varepsilon}{p_0}})(x), Q(x)L_{\frac{\varepsilon}{q_0}}(x))_{\mathbb{C}^m} dx \\ &= \left(\frac{\varepsilon}{p_0 q_0} \right)^{-\frac{d}{2}} \int_{\mathbb{R}^d} (M(\xi), b)_{\mathbb{C}^m} e^{-\frac{-p_0 \pi |\xi - j|^2}{\varepsilon}} e^{-\frac{-q_0 \pi |\xi - k|^2}{\varepsilon}} d\xi =: I \end{aligned}$$

We get that $I \rightarrow \delta_{jk}$ as $\varepsilon \rightarrow 0$ proving the claim.

Now we consider P and Q as trigonometric polynomials, not just the particular monomials used in the proof of the claim. Let $\frac{1}{p} + \frac{1}{q} = 1$ and now we can estimate and get

$$\begin{aligned}
& \left| \int_{\mathbb{T}^n} ((T_{\tilde{M}}P)(x), Q(x)) dx \right| \\
&= \left| \lim_{\varepsilon \rightarrow 0} \varepsilon^{\frac{d}{2}} \int_{\mathbb{R}^d} (T_M(PL_{\frac{\varepsilon}{p_0}})(x), Q(x))_{\mathbb{C}^m} L_{\frac{\varepsilon}{q_0}}(x) dx \right| \\
&\leq \limsup_{\varepsilon \rightarrow 0} \varepsilon^{\frac{d}{2}} \|T_M(PL_{\frac{\varepsilon}{p_0}})\|_{L^{p_0}(\mathbb{R}^d)} \|QL_{\frac{\varepsilon}{q_0}}\|_{L^{q_0}(\mathbb{R}^d)} \\
&\leq \limsup_{\varepsilon \rightarrow 0} \|T_M\| \left\| \varepsilon^{\frac{d}{2p_0}} PL_{\frac{\varepsilon}{p_0}} \right\|_{L^p(\mathbb{R}^d)} \left\| \varepsilon^{\frac{d}{2q_0}} QL_{\frac{\varepsilon}{q_0}} \right\|_{L^{q_0}(\mathbb{R}^d)} \\
&= \|T_M\| \|P\|_{L^p(\mathbb{T}^d)} \|Q\|_{L^{q_0}(\mathbb{T}^d)},
\end{aligned}$$

The last line uses the fact that for a continuous function g the Poisson summation formula gives

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{\frac{dp}{2p_0}} \int_{\mathbb{R}^d} g(x) e^{-\frac{\varepsilon p \pi |x|^2}{p_0}} dx = \int_{\mathbb{T}^d} g(x) dx$$

and similarly with q, q_0 replacing p, p_0 . Note that $\|T_M\| = \|T_M\|_{L^p(\mathbb{R}^d, \mathbb{C}) \rightarrow L^{p_0}(\mathbb{R}^d, \mathbb{C}^m)}$. \square

Lemma 4. Suppose $1 < p \leq p_0 < \infty$ and $M \in \mathcal{V}_m^{p, p_0}(\mathbb{R}^d)$, then $\|T_M\|_{L^p(\mathbb{R}^d, \mathbb{C}) \rightarrow L^{p_0}(\mathbb{R}^d, \mathbb{C}^m)} \leq \|T_{\tilde{M}}\|_{L^p(\mathbb{T}^d, \mathbb{C}) \rightarrow L^{p_0}(\mathbb{T}^d, \mathbb{C}^m)}$.

The proof of this is a similar modification to the scalar case (see pp. 223 – 225 in [8]) as was just shown in Lemma 3.

2.4. Lower Bound of Vector-Valued Multipliers. Let $Q = \mathbb{T}^d$ and \mathcal{P}_k be the trigonometric polynomials from Q^k to \mathbb{C} . The following two theorems are generalizations of the results in [7]. We give a slightly different argument to both that also require some modifications in the more general setting.

Theorem 5. Let $M \in \mathcal{V}_m^{p, p_0}(\mathbb{R}^d)$ be the multiplier corresponding to T_M and

$$T_M^k \left(\sum_{j_1 \in \mathbb{Z}^n} \dots \sum_{j_k \in \mathbb{Z}^n} x_{j_1, \dots, j_k} e^{i(j_1, \theta_1)} \dots e^{i(j_k, \theta_k)} \right) := \sum_{j_1 \in \mathbb{Z}^n} \dots \sum_{j_k \in \mathbb{Z}^n} M(j_k) x_{j_1, \dots, j_k} e^{i(j_1, \theta_1)} \dots e^{i(j_k, \theta_k)}$$

If $\Phi_k \in E_k = \text{clos}_{L^p} \{f \in \mathcal{P}_k : \int_Q f(\theta_1, \dots, \theta_k) d\theta_k = 0\}$ and $1 < p_0 \leq p < \infty$, then

$$\left\| \sum_{k=1}^N T_M^k \Phi_k(\theta_1, \dots, \theta_k) \right\|_{L^{p_0}(Q^N)} \leq \|T_{\tilde{M}}\|_{L^p(Q, \mathbb{C}) \rightarrow L^{p_0}(Q, \mathbb{C}^m)} \left\| \sum_{k=1}^N \Phi_k(\theta_1, \dots, \theta_k) \right\|_{L^p(Q^N)}$$

for all $\Phi_1 \in E_1, \dots, \Phi_N \in E_N$

Proof. Let $\eta \in Q$, f_k a polynomial in E_k and $N \in \mathbb{Z}_+$. For notation we define $f_\eta^k(\theta_1, \dots, \theta_k) := f_k(\theta_1 + N\eta, \dots, \theta_k + N^k\eta)$. Then for all $J \in \mathbb{Z}_+$,

$$\begin{aligned} & \int_Q \left\| \sum_{k=1}^J f_\eta^k \right\|_{L^p(Q^J)}^p \frac{d\eta}{(2\pi)^d} \\ &= \int_Q \int_Q \cdots \int_Q \left\| \sum_{k=1}^J f_k(\theta_1 + N\eta, \dots, \theta_k + N^k\eta) \right\|_{\mathbb{C}^m}^p \frac{d\theta_1}{(2\pi)^d} \cdots \frac{d\theta_J}{(2\pi)^d} \frac{d\eta}{(2\pi)^d} \quad (2.2) \\ &= \int_Q \int_Q \cdots \int_Q \left\| \sum_{k=1}^J f_k(u_1, \dots, u_k) \right\|_{\mathbb{C}^m}^p \frac{du_1}{(2\pi)^d} \cdots \frac{du_J}{(2\pi)^d} \frac{d\eta}{(2\pi)^d} \\ &= \left\| \sum_{k=1}^J f_k(u_1, \dots, u_k) \right\|_{L^p(Q^J)}^p \end{aligned}$$

Similarly,

$$\int_Q \left\| \sum_{k=1}^J T_{\tilde{M}}^k f_\eta^k(\theta_1, \dots, \theta_k) \right\|_{L^{p_0}(Q^J)}^{p_0} \frac{d\eta}{(2\pi)^d} = \left\| \sum_{k=1}^J T_{\tilde{M}}^k f_k(u_1, \dots, u_k) \right\|_{L^{p_0}(Q^J)}^{p_0} \quad (2.3)$$

For fixed $\vec{\theta} = (\theta_1, \dots, \theta_k)$ we will use the notation $F_{\vec{\theta}}^k(\eta) := f_\eta^k(\theta_1, \dots, \theta_k)$ to emphasize that we are thinking of the quantity as a function of η and $\vec{\theta}$ as a parameter. This is because $T_{\tilde{M}}$ can only act on a function of one variable $\eta \in Q$. Also, from this point on in the proof, all measures will be normalized Lebesgue measures. We claim that

$$\lim_{N \rightarrow \infty} \left| \int_Q \left\| \sum_{k=1}^J T_{\tilde{M}}^k f_\eta^k(\theta_1, \dots, \theta_k) \right\|_{L^{p_0}(Q^J, \vec{\theta})}^{p_0} d\eta - \int_{Q^J} \left\| \sum_{k=1}^J T_{\tilde{M}}^k F_{\vec{\theta}}^k(\eta) \right\|_{L^{p_0}(Q, \eta)}^{p_0} d\vec{\theta} \right| = 0 \quad (2.4)$$

with convergence being uniform. Let us prove this claim.

Since $f_\eta^k(\theta_1, \dots, \theta_k) = \sum_{l_1 \in \mathbb{Z}^n} \cdots \sum_{l_k \in \mathbb{Z}^n} x_{l_1, \dots, l_k} e^{i(l_1, \theta_1 + N\eta)} \cdots e^{i(l_k, \theta_k + N^k\eta)}$ then

$$T_{\tilde{M}}^k f_\eta^k(\theta_1, \dots, \theta_k) = \sum_{l_1 \in \mathbb{Z}^n} \cdots \sum_{l_k \in \mathbb{Z}^n} M(l_k) x_{l_1, \dots, l_k} e^{i(l_1, \theta_1 + N\eta)} \cdots e^{i(l_k, \theta_k + N^k\eta)}$$

and $T_{\tilde{M}}^k F_{\vec{\theta}}^k(\eta) = \sum_{l_1 \in \mathbb{Z}^n} \cdots \sum_{l_k \in \mathbb{Z}^n} M(Nl_1 + \cdots + N^k l_k) x_{l_1, \dots, l_k} e^{i(l_1, \theta_1 + N\eta)} \cdots e^{i(l_k, \theta_k + N^k\eta)}$ But, $M(Nl_1 + \cdots + N^k l_k) = M(l_k + \frac{1}{N}l_{k-1} + \cdots + \frac{1}{N^{k-1}}l_1) \rightarrow M(l_k)$ as $N \rightarrow \infty$. Note that $l_k \neq 0$ by how we defined E_k , so the fact that M may not be continuous at the origin is fine. Then,

$$\left(\int_Q \left\| \sum_{k=1}^J T_{\tilde{M}}^k f_\eta^k(\theta_1, \dots, \theta_k) \right\|_{L^{p_0}(Q^J, \vec{\theta})}^{p_0} d\eta \right)^{\frac{1}{p_0}} - \left(\int_{Q^J} \left\| \sum_{k=1}^J T_{\tilde{M}}^k F_{\vec{\theta}}^k(\eta) \right\|_{L^{p_0}(Q, \eta)}^{p_0} d\vec{\theta} \right)^{\frac{1}{p_0}}$$

$$\begin{aligned}
&= \left(\int_Q \left\| \sum_{k=1}^J T_{\tilde{M}}^k f_{\eta}^k(\theta_1, \dots, \theta_k) \right\|_{L^{p_0}(Q^J, \vec{\theta})}^{p_0} d\eta \right)^{\frac{1}{p_0}} - \left(\int_Q \left\| \sum_{k=1}^J T_{\tilde{M}} F_{\theta}^k(\eta) \right\|_{L^{p_0}(Q^J, \vec{\theta})}^{p_0} d\eta \right)^{\frac{1}{p_0}} \\
&\leq \left(\int_Q \left\| \sum_{k=1}^J T_{\tilde{M}}^k f_{\eta}^k(\theta_1, \dots, \theta_k) - T_{\tilde{M}} F_{\theta}^k(\eta) \right\|_{L^{p_0}(Q^J, \vec{\theta})}^{p_0} d\eta \right)^{\frac{1}{p_0}} \\
&\leq \left(\sum_{k=1}^J \sum_{l_1 \in \mathbb{Z}^n} \dots \sum_{l_k \in \mathbb{Z}^n} \|x_{l_1, \dots, l_k}\|_{\mathbb{C}^m}^{p_0} \|M(Nl_1 + \dots + N^k l_k) - M(l_k)\|^{p_0} \right)^{\frac{1}{p_0}} \xrightarrow{N \rightarrow \infty} 0
\end{aligned}$$

This completes the proof of the claim.

Now we just need some easy estimates to finish the proof of the Theorem. For notation, let $\vec{u} = (u_1, \dots, u_J) \in Q^J$ and du_i the normalized Lebesgue measure on \mathbb{T}^d . Then for sufficiently large N and a given $\varepsilon > 0$, we can use (2.2), (2.3) and (2.4) to get

$$\begin{aligned}
&\left\| \sum_{k=1}^J T_{\tilde{M}}^k f_k(u_1, \dots, u_k) \right\|_{L^{p_0}(Q^J, \vec{u})}^{p_0} \\
&= \int_Q \left\| \sum_{k=1}^J T_{\tilde{M}}^k f_{\eta}^k(\theta_1, \dots, \theta_k) \right\|_{L^{p_0}(Q^J, \vec{u})}^{p_0} d\eta \\
&\leq \int_{Q^J} \left\| \sum_{k=1}^J T_{\tilde{M}} F_{\theta}^k(\eta) \right\|_{L^{p_0}(Q, \eta)}^{p_0} d\vec{\theta} + \varepsilon \\
&\leq \|T_{\tilde{M}}\|_{L^p(Q) \rightarrow L^{p_0}(Q)}^{p_0} \int_{Q^J} \left\| \sum_{k=1}^J F_{\theta}^k(\eta) \right\|_{L^p(Q, \eta)}^{p_0} d\vec{\theta} + \varepsilon \\
&= \|T_{\tilde{M}}\|_{L^p(Q) \rightarrow L^{p_0}(Q)}^{p_0} \int_{Q^J} \left(\int_Q \left\| \sum_{k=1}^J F_{\theta}^k(\eta) \right\|_{\mathbb{C}^m}^p d\eta \right)^{\frac{p_0}{p}} d\vec{\theta} + \varepsilon \\
&\leq \|T_{\tilde{M}}\|_{L^p(Q) \rightarrow L^{p_0}(Q)}^{p_0} \left(\int_{Q^J} \int_Q \left\| \sum_{k=1}^J F_{\theta}^k(\eta) \right\|_{\mathbb{C}^m}^p d\eta d\vec{\theta} \right)^{\frac{p_0}{p}} + \varepsilon \tag{2.5} \\
&= \|T_{\tilde{M}}\|_{L^p(Q) \rightarrow L^{p_0}(Q)}^{p_0} \left(\int_{Q^J} \left\| \sum_{k=1}^J f_{\eta}^k(\theta_1, \dots, \theta_k) \right\|_{L^p(Q, \eta)}^p d\vec{\theta} \right)^{\frac{p_0}{p}} + \varepsilon \\
&= \|T_{\tilde{M}}\|_{L^p(Q) \rightarrow L^{p_0}(Q)}^{p_0} \left\| \sum_{k=1}^J f_k(u_1, \dots, u_k) \right\|_{L^p(Q^J, \vec{u})}^{p_0} + \varepsilon
\end{aligned}$$

Note that we used the concave version of Jensen's inequality in (2.5). \square

Theorem 6. Fix $\tau \in \mathbb{R}$ and let $M = \begin{pmatrix} m \\ \tau \end{pmatrix}$ such that $m : \mathbb{R}^d \rightarrow \mathbb{R}$ be even, continuous (except possibly at the origin) and homogeneous of order zero multiplier. If there exists $v^\pm \in S^{d-1}$ (the unit sphere) such that $m(v^\pm) = \delta^\pm$, $\delta^+ = -\delta^- > 0$ then $\|T_M\|_{L^p(\mathbb{R}^d, \mathbb{C}) \rightarrow L^{p_0}(\mathbb{R}^d, \mathbb{C}^2)} \geq \frac{2C_{p,p_0}^\tau}{\delta^+ - \delta^-}$.

Proof. For this proof we will continue using the notation Q to represent \mathbb{T}^d . Let $\varepsilon > 0$ be given. By the definition of C_{p,p_0}^τ , there exists $\vec{\beta} = (\beta_1, \dots, \beta_N)$ and complex-valued $d_1(r_0), d_2(r_0, r_1), \dots, d_N(r_0, \dots, r_{N-1})$, where $\beta_k \in \{\pm 1\}$ such that

$$C_{p,p_0}^\tau - \frac{\left\| \sum_{k=1}^N \begin{pmatrix} \beta_k \\ \tau \end{pmatrix} r_k d_k(r_0, \dots, r_{k-1}) \right\|_{p_0}}{\left\| \sum_{k=1}^N r_k d_k(r_0, \dots, r_{k-1}) \right\|_p} \leq \varepsilon \quad (2.6)$$

Let $\delta > 0$ be given and $m(v^\pm) = \delta^\pm$, then there exists $\{n_k^\pm\} \subset \mathbb{Z}^d$ such that $\|m(n_k^\pm) - m(v^\pm)\| < \delta$ for k sufficiently large. To ease notation, let n^\pm be one such n_k^\pm . Since sign is an odd L^p function on $[-\pi, \pi)$ then $\text{sign}(\theta) = \sum_{j \in \mathbb{Z} \setminus \{0\}} c_j \sin(j\theta)$ for some c_j 's, where $\theta \in [-\pi, \pi)$. Let $\psi^\pm(\theta) = \text{sign}(n^\pm, \theta)$, $\theta \in Q$ and $\{\alpha_k\}_{k=0}^N$ an arbitrary sequence from $\{\delta^\pm\}$. Then if we define

$$\psi_k := \begin{cases} \psi^+ & \alpha_k = \delta^+ \\ \psi^- & \alpha_k = \delta^- \end{cases}$$

an easy computation shows that

$$T_M^k[\psi_k(\theta_k) d_k(\psi_0(\theta_0), \dots, \psi_{k-1}(\theta_{k-1}))] = \begin{cases} M(n^+) \psi_k(\theta_k) d_k(\psi_0(\theta_0), \dots, \psi_{k-1}(\theta_{k-1})) & , \alpha_k \in \delta^+ \\ M(n^-) \psi_k(\theta_k) d_k(\psi_0(\theta_0), \dots, \psi_{k-1}(\theta_{k-1})) & , \alpha_k \in \delta^- \end{cases}$$

Since (r_0, \dots, r_N) has the same distribution on $[0, 1)$ as $(\psi_0(\theta_1), \dots, \psi_N(\psi_N))$ has on Q^N then

$$\begin{aligned}
& \left\| \sum_{k=1}^N \begin{pmatrix} \alpha_k \\ \tau \end{pmatrix} r_k d_k(r_0, \dots, r_{k-1}) \right\|_{L^{p_0}[0,1)} \\
&= \left\| \sum_{k=1}^N \begin{pmatrix} \alpha_k \\ \tau \end{pmatrix} \psi_k(\theta_k) d_k(\psi_0(\theta_0), \dots, \psi_{k-1}(\theta_{k-1})) \right\|_{L^{p_0}(Q^N)} \\
&\leq \left\| \sum_{k=1}^N M((n^+ \vee n^-)_k) \psi_k(\theta_k) d_k(\psi_0(\theta_0), \dots, \psi_{k-1}(\theta_{k-1})) \right\|_{L^{p_0}(Q^N)} \\
&\quad + \left\| \sum_{k=1}^N \left[\begin{pmatrix} \alpha_k \\ \tau \end{pmatrix} - M((n^+ \vee n^-)_k) \right] \psi_k(\theta_k) d_k(\psi_0(\theta_0), \dots, \psi_{k-1}(\theta_{k-1})) \right\|_{L^{p_0}(Q^N)} \\
&=: I + II
\end{aligned}$$

The notation

$$M((n^+ \vee n^-)_k) = \begin{cases} M(n^+) & , \text{ if } \alpha_k = \delta^+ \\ M(n^-) & , \text{ if } \alpha_k = \delta^- \end{cases}$$

and we similarly define $M((v^+ \vee v^-)_k)$. Now,

$$\begin{aligned}
II &= \left\| \sum_{k=1}^N [m((v^+ \vee v^-)_k) - m((n^+ \vee n^-)_k)] \psi_k(\theta_k) d_k(\psi_0(\theta_0), \dots, \psi_{k-1}(\theta_{k-1})) \right\|_{L^{p_0}(Q^N)} \\
&\leq \delta \sum_{k=1}^N \|\psi_k(\theta_k) d_k(\psi_0(\theta_0), \dots, \psi_{k-1}(\theta_{k-1}))\|_{L^{p_0}(Q^N)} \xrightarrow{\delta \rightarrow 0} 0
\end{aligned}$$

Using Theorem 5 gives

$$\begin{aligned}
I &= \left\| \sum_{k=1}^N T_{\widetilde{M}}^k [\psi_k(\theta_k) d_k(\psi_0(\theta_0), \dots, \psi_{k-1}(\theta_{k-1}))] \right\|_{L^{p_0}(Q^N)} \\
&\leq \|T_{\widetilde{M}}\|_{L^p(Q, \mathbb{C}) \rightarrow L^{p_0}(Q, \mathbb{C}^2)} \left\| \sum_{k=1}^N \psi_k(\theta_k) d_k(\psi_0(\theta_0), \dots, \psi_{k-1}(\theta_{k-1})) \right\|_{L^p(Q^N)} \\
&= \|T_{\widetilde{M}}\|_{L^p(Q, \mathbb{C}) \rightarrow L^{p_0}(Q, \mathbb{C}^2)} \left\| \sum_{k=1}^N r_k d_k(r_0, \dots, r_{k-1}) \right\|_{L^p[0,1)}
\end{aligned}$$

So, all together we have

$$\begin{aligned}
 & \left\| \sum_{k=1}^N \begin{pmatrix} \alpha_k \\ \tau \end{pmatrix} r_k d_k(r_0, \dots, r_{k-1}) \right\|_{L^{p_0}[0,1]} \\
 & \leq \|T_{\tilde{M}}\|_{L^p(Q, \mathbb{C}) \rightarrow L^{p_0}(Q, \mathbb{C}^2)} \left\| \sum_{k=1}^N r_k d_k(r_0, \dots, r_{k-1}) \right\|_{L^p[0,1]} \quad (2.7)
 \end{aligned}$$

Now that we have our main estimate we just need to rescale δ^\pm to ± 1 to get C_{p,p_0}^τ into the estimate. Let $A = \frac{2}{\delta^+ - \delta^-}$, then we can choose the correct α_k so that

$$A\alpha_k =: \beta_k = \begin{cases} +1, & \alpha_k = \delta^+ \\ -1, & \alpha_k = \delta^-, \end{cases}$$

where β_k satisfies the estimate in (2.6). We can use (2.6) and (2.7) to get

$$\begin{aligned}
 C_{p,p_0}^\tau & \leq \frac{\left\| \sum_{k=1}^N \begin{pmatrix} \beta_k \\ \tau \end{pmatrix} r_k d_k(r_0, \dots, r_{k-1}) \right\|_{p_0}}{\left\| \sum_{k=1}^N r_k d_k(r_0, \dots, r_{k-1}) \right\|_p} + \varepsilon \\
 & \leq |A| \|T_{\tilde{M}}\|_{L^p(Q, \mathbb{C}) \rightarrow L^{p_0}(Q, \mathbb{C}^2)} + \varepsilon \\
 & \leq |A| \|T_M\|_{L^p(\mathbb{R}^d, \mathbb{C}) \rightarrow L^{p_0}(\mathbb{R}^d, \mathbb{C}^2)} + \varepsilon
 \end{aligned}$$

where the last inequality is due to Theorem 3. □

Corollary 7. Fix $\tau \in \mathbb{R}$ and let $M = \begin{pmatrix} m \\ \tau \end{pmatrix}$, where m is the multiplier corresponding to the real or imaginary part of the Ahlfors-Beurling transform. Then m takes a maximum and minimum value of ± 1 on the unit circle and therefore $\|T_M\|_{L^p(\mathbb{C}, \mathbb{C}) \rightarrow L^p(\mathbb{C}, \mathbb{C}^2)} \geq C_{p,p}^\tau$.

Note that $C_{p,p}^\tau$ was computed as $((p^* - 1)^2 + \tau^2)^{\frac{1}{2}}$, for $|\tau| \leq \frac{1}{2}$ and $1 < p < 2$ or $\tau \in \mathbb{R}$ and $2 \leq p < \infty$ in [1], [2] and [3]. For a definition and properties of the Ahlfors-Beurling operator look ahead to Section 3.5.

Remark 8. We expect the converse inequality to be the same. So this lower bound technique will be sharp just as when $\tau = 0$. This tells us that if we are able to determine the operator norm of some perturbation of the martingale transform, then we will be able to determine the lower bound of the same perturbation of $\Re B$ and $\Im B$, using the same proof as here.

3. Matrix-Valued Multipliers

3.1. Multipliers. Generalizing the idea of a scalar multiplier, let us consider $M = \{m_{ij}\}_{1 \leq i, j \leq m}$ as a matrix-valued function whose entries are bounded functions from \mathbb{R}^n to \mathbb{C} and p, p_0 such that $1 \leq p, p_0 \leq \infty$. We say that M is the multiplier corresponding to the operator T_M , if $T_M f := (M\widehat{f})^\vee$ is a bounded operator from $L^p(\mathbb{R}^n, \mathbb{C}^m)$ to $L^{p_0}(\mathbb{R}^n, \mathbb{C}^m)$, considering functions f from \mathbb{R}^n to \mathbb{C}^m with components in \mathcal{S} . Note that the Fourier transform of f is defined componentwise and $\|f\|_{L^p(\mathbb{R}^n, \mathbb{C}^m)}^p := \int_{\mathbb{R}^n} \|f(x)\|_{\mathbb{C}^m}^p dx$. We use the notation $\widetilde{M} = (\widetilde{m}_{ij})_{1 \leq i, j \leq m}$ to denote the discrete multiplier.

Remark 9. We will use the notation $\mathcal{M}_{m \times n}^{p, p_0}(\mathbb{R}^d)$ to denote the class of all $m \times n$ multipliers whose components are continuous (except possibly at zero) and homogeneous of order zero, where $\|M\|_{\mathcal{M}_{m \times n}^{p, p_0}} := \|T_M\|_{L^p(\mathbb{R}^d, \mathbb{C}^n) \rightarrow L^{p_0}(\mathbb{R}^d, \mathbb{C}^m)}$ for any such $M \in \mathcal{M}_{m \times n}^{p, p_0}(\mathbb{R}^d)$.

3.2. Generalizing Unconditional Martingale Differences. The idea of an Unconditional Martingale Difference (UMD) will be generalized here. Let X be a Banach space and $\{r_n\}_{n \geq 0}$ the Rademacher sequence. Let $\{F_N\}_{N \geq 0}$ and $\{G_N\}_{N \geq 0}$ be X -valued martingale difference sequences of the form

$$F_N = \sum_{k=1}^N d_k(r_0, \dots, r_{k-1}) r_k, \quad G_N = \sum_{k=1}^N \beta_k d_k(r_0, \dots, r_{k-1}) r_k$$

where $d_k : \{\pm 1\}^k \rightarrow X$ is Bochner integrable and $\vec{\beta}$ is a vector with entries $\beta_i \in \pm 1$. For any particular such $\vec{\beta}$, $MT_{\vec{\beta}}(F_N) := G_N$ is a martingale transform.

Definition 10. $UMD_{p, p_0}(X) := \sup_{\vec{\beta}} \|MT_{\vec{\beta}}\|_{L_X^p[0,1) \rightarrow L_X^{p_0}[0,1)}$

Note that $UMD_p(X) = UMD_{p,p}(X)$. For definitions and properties regarding $UMD_p(X)$ for a Banach space X , refer to [5]. There is one property that is very useful in getting sharp estimates for singular integrals. The property is that $UMD_p(\mathcal{H}) = p^* - 1$, where \mathcal{H} is a Hilbert space.

3.3. Transference. The same proofs from Section 2.3 apply here to the following.

Lemma 11. Suppose $1 < p_0 \leq p < \infty$ and $M \in \mathcal{M}_{m \times n}^{p, p_0}(\mathbb{R}^d)$, then $\|T_M\|_{L^p(\mathbb{R}^d, \mathbb{C}^n) \rightarrow L^{p_0}(\mathbb{R}^d, \mathbb{C}^m)} \leq \|T_{\widetilde{M}}\|_{L^p(\mathbb{T}^d, \mathbb{C}^n) \rightarrow L^{p_0}(\mathbb{T}^d, \mathbb{C}^m)}$.

Lemma 12. Suppose $1 < p \leq p_0 < \infty$ and $M \in \mathcal{M}_{m \times n}^{p, p_0}(\mathbb{R}^d)$, then $\|T_M\|_{L^p(\mathbb{R}^d, \mathbb{C}^n) \rightarrow L^{p_0}(\mathbb{R}^d, \mathbb{C}^m)} \leq \|T_{\widetilde{M}}\|_{L^p(\mathbb{T}^d, \mathbb{C}^n) \rightarrow L^{p_0}(\mathbb{T}^d, \mathbb{C}^m)}$.

3.4. Lower Bound of Matrix-valued Multipliers. Let \mathcal{P}_k be a trigonometric polynomial taking k variables in $Q := \mathbb{T}^d$. The following Theorems have nearly identical proofs to those in Section 2.4 so we omit the details.

Theorem 13. Let $M \in \mathcal{M}_{m \times n}^{p, p_0}(\mathbb{R}^d)$, corresponding to T_M and

$$T_M^k \left(\sum_{j_1 \in \mathbb{Z}^n} \dots \sum_{j_k \in \mathbb{Z}^n} x_{j_1, \dots, j_k} e^{i(j_1, \theta_1)} \dots e^{i(j_k, \theta_k)} \right) := \sum_{j_1 \in \mathbb{Z}^n} \dots \sum_{j_k \in \mathbb{Z}^n} M(j_k) x_{j_1, \dots, j_k} e^{i(j_1, \theta_1)} \dots e^{i(j_k, \theta_k)}$$

If $\Phi_k \in E_k = \text{clos}_{L^p} \{ \mathcal{P}_k : \int_Q \mathcal{P}_k(\theta_1, \dots, \theta_k) d\theta_k = 0 \}$ and $p \geq p_0 > 1$, then

$$\left\| \sum_{k=1}^N T_M^k \Phi_k(\theta_1, \dots, \theta_k) \right\|_{L^{p_0}(Q^d, \mathbb{C}^m)} \leq \|T_M\|_{L^p(Q, \mathbb{C}^n) \rightarrow L^{p_0}(Q, \mathbb{C}^m)} \left\| \sum_{k=1}^N \Phi_k(\theta_1, \dots, \theta_k) \right\|_{L^p(Q^d, \mathbb{C}^n)}$$

for all $\Phi_1 \in E_1, \dots, \Phi_N \in E_N$

For $1 < p_0 \leq p < \infty$, denote $\mathcal{M}_{p, p_0}(\mathbb{M}_m)$ as the collection of $m \times m$ matrix valued multipliers whose components are real, even, continuous (except possibly at zero) and homogeneous order zero functions in \mathbb{R}^n with $\|M\|_{\mathcal{M}_{p, p_0}(\mathbb{R}^n)} := \|T_M\|_{L^p \rightarrow L^{p_0}}$. We will use the notation $\mathcal{M}_p(\mathbb{M}_m)$ for $\mathcal{M}_{p, p}(\mathbb{M}_m)$.

Theorem 14. Suppose $M \in \mathcal{M}_{p, p_0}(\mathbb{M}_m)$. If there exists $v^\pm \in \mathbb{R}^n$ such that $M(v^\pm) = \delta^\pm U$, where U is a unitary matrix and $\delta^+ > \delta^-$, then

$$\|T_M\|_{L^p(\mathbb{R}^n, \mathbb{C}^m) \rightarrow L^{p_0}(\mathbb{R}^n, \mathbb{C}^m)} \geq \|T_{\tilde{M}}\|_{L^p(\mathbb{T}^n, \mathbb{C}^m) \rightarrow L^{p_0}(\mathbb{T}^n, \mathbb{C}^m)} \geq \frac{UMD_{p, p_0}(\mathbb{C}^m)}{\frac{2}{\delta^+ - \delta^-} \left(1 + \frac{|\delta^+ + \delta^-|}{|\delta^+| + |\delta^-|} \right)}.$$

Remark 15. The main application of Theorem 14 will be to matrices M that are unitarily diagonalizable and satisfy the needed assumptions. Recall, for example, that self-adjoint matrices are unitarily diagonalizable. Also, a 2×2 matrix that is the sum of a skew-symmetric matrix and diagonal matrix is unitarily diagonalizable as well, as we will use as an application in the next section.

Remark 16. Notice how in Theorem 14 if we use a multiplier M that is nearly the identity matrix, but still satisfying all of the assumptions, then the estimate is quite bad, even though $\|T_M\|$ should be approximately 1. This suggests that there is room for improvement with our result.

3.5. Applications. For the Ahlfors-Beurling operator defined as

$$Bf(z) := p.v. - \frac{1}{\pi} \int_{\mathbb{C}} \frac{f(w)}{(z-w)^2} dm_2(w),$$

it is known that $\|B\|_p := \|B\|_{L^p(\mathbb{C}) \rightarrow L^p(\mathbb{C})} \geq p^* - 1$, for all p such that $1 < p < \infty$, as originally shown by Lehto in [9] in 1965. There has not been (to the authors knowledge) an alternate proof of Lehto's result, but it now follows as an easy application of Theorem 14.

Corollary 17. $\|B\|_p \geq p^* - 1$, for all p such that $1 < p < \infty$

Proof. It is known that $-B = R_2^2 - R_1^2 + 2iR_1R_2$ (R_1 and R_2 are the planar Riesz transforms) and the multiplier corresponding to B is $m(\xi) = \frac{\xi_2^2 - \xi_1^2 + 2i\xi_1\xi_2}{|\xi|^2} =: m_R(\xi) + im_I(\xi)$. Let $M = \begin{pmatrix} m_R(\xi) & m_I(\xi) \\ -m_I(\xi) & m_R(\xi) \end{pmatrix}$ and $\widehat{f} = u + iv$. Since we have that $m(\xi)\widehat{f} = M(\xi) \begin{pmatrix} u \\ v \end{pmatrix}$, then there exists an isomorphism taking $(m\widehat{f})^\vee$ to $(M\widehat{f})^\vee$ and preserving the norm. So we can conclude that $\|B\|_p = \|T_M\|_{L^p(\mathbb{C}) \rightarrow L^p(\mathbb{R}^2)}$. Observe that $m_R(1,0) = 1$ and $m_R(0,1) = -1$ are the maximum and minimum of m_R on $\partial B(0,1)$ respectively. Theorem 14 implies that $\|T_M\|_{L^p(\mathbb{C}) \rightarrow L^p(\mathbb{R}^2)} \geq p^* - 1$. \square

Corollary 18. $\|(R_1^2 - R_2^2)\cos(\theta) + 2R_1R_2\sin(\theta)\|_{L^p(\mathbb{C}) \rightarrow L^p(\mathbb{C})} = p^* - 1$ for all θ and all $1 < p < \infty$

Proof. If $|\xi| = 1$ then the multiplier corresponding to the operator is given by $(\xi_1^2 - \xi_2^2)\cos\theta + 2\xi_1\xi_2\sin\theta$ and can be written as $\cos(2\phi - \theta)$ by letting $\xi_1 = \cos\phi$ and $\xi_2 = \sin\phi$. So the multiplier will achieve its maximum and minimum of 1 and -1 on the unit circle. By Theorem 14 we have the lower bound of the operator is $p^* - 1$. The upper bound comes from [6]. \square

Assumption 19. For the remainder of the paper let us assume that the Iwaniec conjecture ($\|B\|_p = p^* - 1$ for all $1 < p < \infty$) is true.

Corollary 20.

$$\|c(R_1^2 - R_2^2) + 2iR_1R_2\|_{L^p(\mathbb{C}) \rightarrow L^p(\mathbb{C})} = \begin{cases} (p^* - 1) & , 0 < |c| < 1 \\ |c|(p^* - 1) & , |c| \geq 1 \end{cases}$$

for all p such that $1 < p < \infty$.

Proof. If $c \geq 1$, then it is easy to see that the operator is bounded below by $c(p^* - 1)$, by simply using the same argument as in Corollary 17. The upper bound for the operator is $(c - 1)\|R_1^2 - R_2^2\|_p + \|B\|_p \leq c(p^* - 1)$ by [10] and since $\|B\|_p = p^* - 1$.

If $0 < c < 1$, then we can write the operator as $c\|(R_1^2 - R_2^2) + 2i\frac{1}{c}R_1R_2\|_p = c\|\frac{1}{c}(R_1^2 - R_2^2) + 2iR_1R_2\|_p$, since $(R_1^2 - R_2^2)$ is unitarily equivalent to $2iR_1R_2$ (using a rotation by $\frac{\pi}{4}$). Applying the case when $c \geq 1$ gives the desired result.

For the cases where c is negative we just need to observe that for multipliers of the form $m(\xi) = \frac{(\xi, A\xi)}{|\xi|^2}$, we have the property $(T_m f)(x) = \overline{(T_{\overline{m}} \overline{f})}(x)$. This yields $\|T_m\|_p = \|T_{\overline{m}}\|_p$. But, $\frac{c(\xi_1^2 - \xi_2^2) + 2i\xi_1\xi_2}{|\xi|^2}$ is the multiplier corresponding the given operator and the conjugate corresponds to the operator $c(R_1^2 - R_2^2) - 2iR_1R_2$. So $\|c(R_1^2 - R_2^2) + 2iR_1R_2\|_p = \|-c(R_1^2 - R_2^2) - 2iR_1R_2\|_p = \|-c(R_1^2 - R_2^2) + 2iR_1R_2\|_p$. Using the two cases that we already have shown will give us the result for c negative. \square

Corollary 21. Let $F(z) := (R_1^2 - R_2^2) + 2zR_1R_2$, then for all p such that $1 < p < \infty$,

$$\|F(z)\|_{L^p(\mathbb{C}) \rightarrow L^p(\mathbb{C})} = \begin{cases} \sqrt{1+z^2}(p^* - 1), & z \in \mathbb{R} \\ (p^* - 1), & \operatorname{Re}(z) = 0, |z| < 1 \\ |z|(p^* - 1), & \operatorname{Re}(z) = 0, |z| \geq 1 \end{cases}$$

Proof. If $z = x \in \mathbb{R}$ then $\|F(z)\|_p = \sqrt{1+x^2} \left\| \frac{1}{\sqrt{1+x^2}}(R_1^2 - R_2^2) + \frac{x}{\sqrt{1+x^2}}2R_1R_2 \right\|_p = \sqrt{1+x^2}(p^* - 1)$ by making the substitution $x = \tan \theta$ and using Corollary 18. The other two cases follow directly from Corollary 20. \square

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NICHOLAS BOROS, DEPT. OF MATH., MICHIGAN STATE UNIVERSITY. borosnic@msu.edu

ALEXANDER VOLBERG, DEPT. OF MATH., MICHIGAN STATE UNIVERSITY. volberg@math.msu.edu